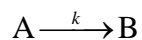


NMR Relaxation

Review of First-Order Rate Kinetics

Imagine the simple first order process:



The time-dependence of the conversion of A to B depends upon the amount of A present according to the following expression:

$$\frac{\Delta[A]}{\Delta t} = -k[A]$$

or in terms of infinitesimally small variations:

$$\frac{d[A]}{dt} = -k[A]$$

This expression may be rearranged to produce the following

$$\frac{d[A]}{A} = -k dt$$

Integration leads to

$$\int \frac{d[A]}{A} = -k \int dt$$

$$\ln[A] = -kt + C$$

$$[A] = [A]_0 e^{-kt}$$

k is known as the rate constant of the first-order process or alternatively as the time-constant of the single-exponential reaction.

Nuclear Induction

If a macroscopic sample of magnetically active nuclei is placed in a static magnetic field, a polarization or macroscopic magnetization will be produced *within the sample*. This phenomenon is known as *nuclear induction*.

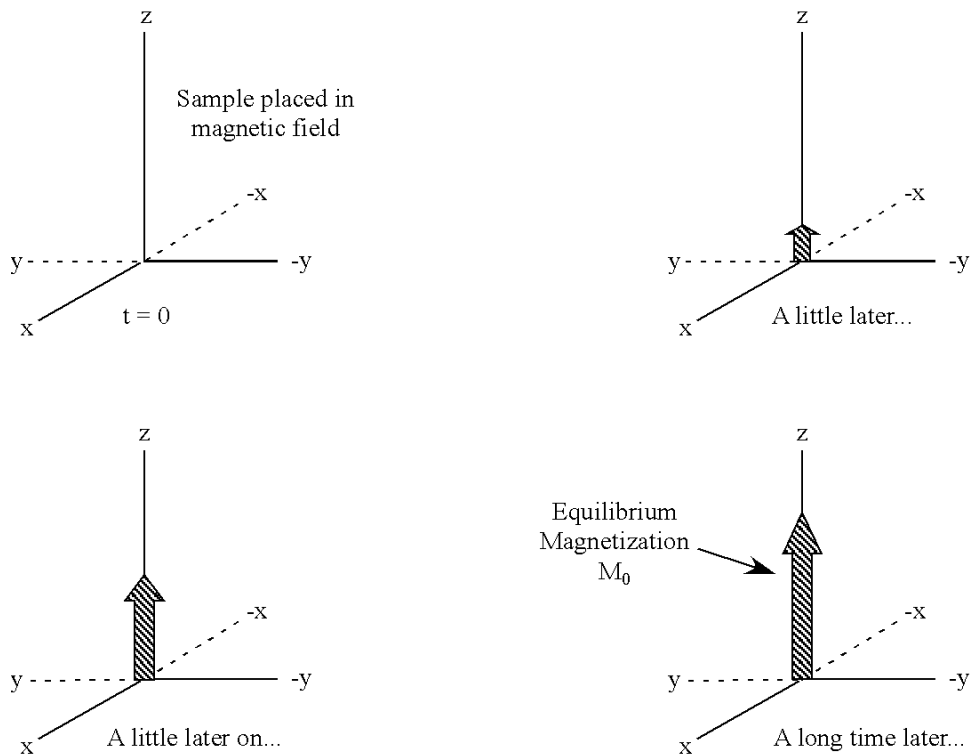
This induced magnetization does not arise instantaneously, but instead accumulates at an exponential rate.

If we define the orientation of the static magnetic field to be along and z-direction, then the induced polarization will also be along the z-axis.

Again, the induced magnetization accumulates at an exponential rate.

$$M(t) \propto 1 - e^{-Rt}$$

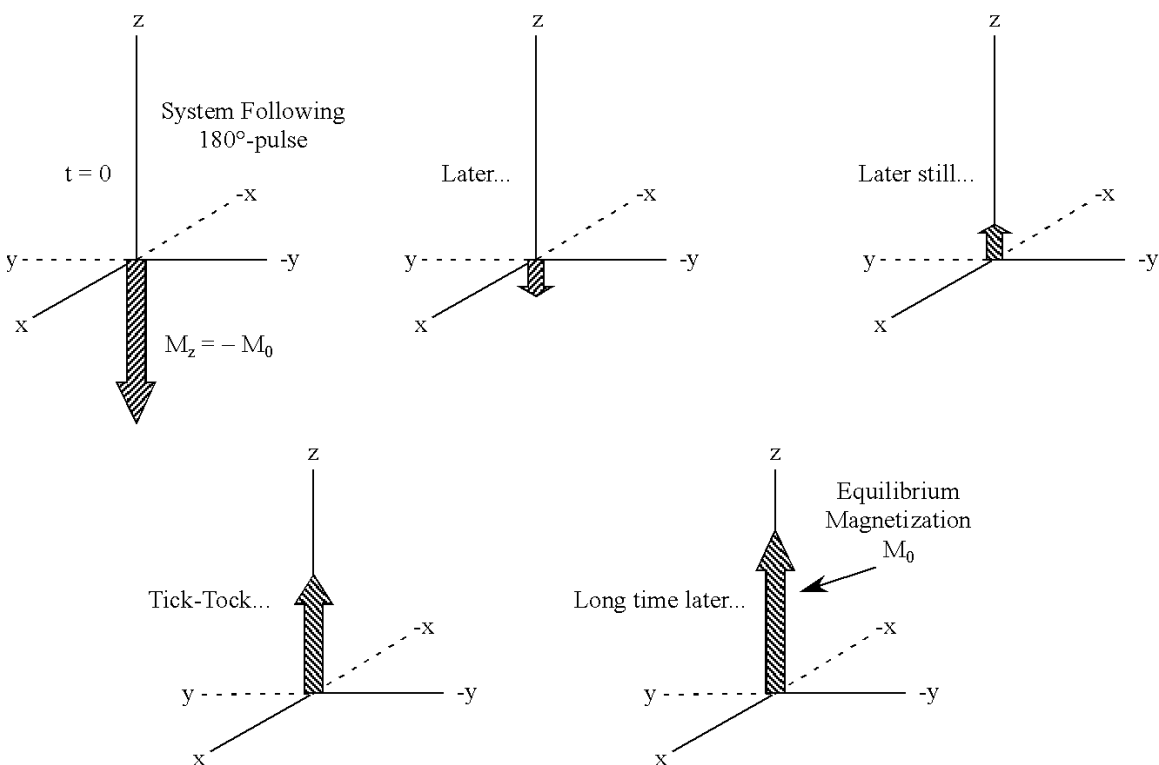
A geometrical picture should help us develop a more intuitive sense of the effect:



Z-axis Recovery from Perturbation

If we allow such a macroscopic magnetization to establish itself and then apply resonant RF irradiation to tip the vector away from the static field axis, and then monitor the magnitude of the magnetization *along the z-axis* we would find that the system will recover, or *relax* back to the original state at exactly the same rate as for the induction experiment.

Again, the geometrical analysis may provide a clearer sense of the physics:



The *rate* of recovery (the relaxation rate) of the magnetization along the z-direction does not depend on the initial state of the system. This means that the relaxation rate along the z-direction does not depend upon the extent of perturbation, i.e., the rate of recovery will not depend on whether we used a 30°- or 60°- or 120° pulses instead of the 180°-pulse.

Note that although the *rate* depends only upon the relative change in the amplitude, as we saw in the analysis of the first-order reaction, the *actual value* of the observable depends upon both the rate and the initial value of the observable.

Relaxation in the XY-Plane

In the preceding example we monitored the relaxation effects following and inversion pulse, a 180° -pulse. Consider an alternative experiment in which we apply a 90° -pulse and again monitor the response of the system.

If we monitor the recovery along the z-direction we would measure the exact same rate as in the previous example.

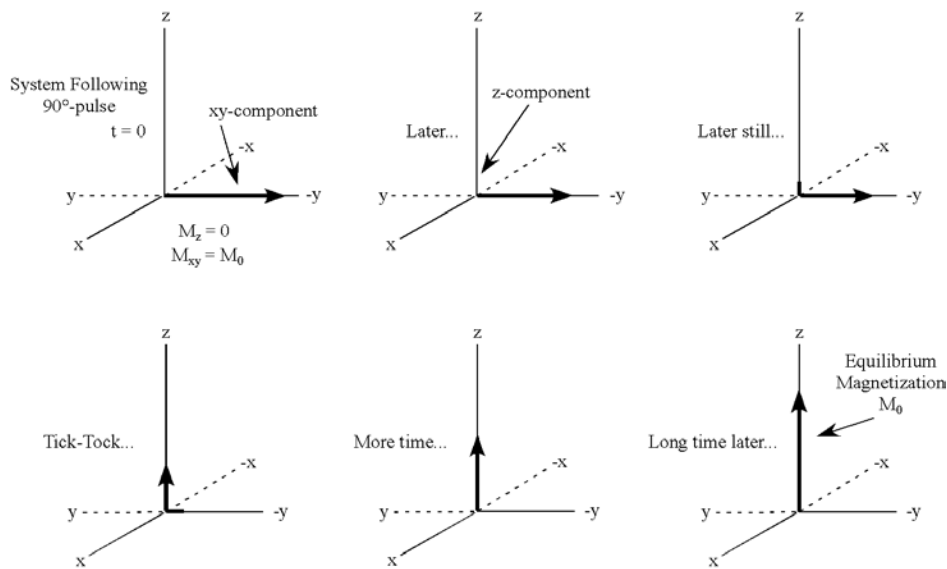
If in this same experiment we were to monitor the amplitude of the magnetization along either the x- or y-directions, we would again find that the rate of change was exponential.

We would also discover that the rate of change along the x-direction was exactly the same as the rate of change along the y-direction.

Finally, we would find that rate of change in the x- or y-directions would not exactly match the rate at which magnetization recovered along the z direction.

Specifically, we would observed that the amplitude of the magnetization decreased at a *faster* rate in the xy-direction that the z-direction.

The results are shown schematically summarized below:



Bloch's Equations

A phenomenological equation is one that is derived to account for experimentally observed behavior. Contrast this with a theoretical equation that expresses predicted behavior based on a hypothetical model.

Dynamics is the study of how a system varies in time.

Bloch generated a series of expressions that account for observed dynamics of the macroscopic magnetization. The resulting expressions are known as Bloch's equations:

$$\frac{d}{dt} \mathbf{M}(t) = \gamma \mathbf{M}(t) \times \mathbf{B}(t) - R [\mathbf{M}(t) - \mathbf{M}_0]$$

\mathbf{M}_0 is the equilibrium value of \mathbf{M} (its value in the absence of perturbations), and the inherent *time-dependence* of the components is emphasized.

$\mathbf{M}(t)$ is a vector in this expression, and may be written as:

$$\mathbf{M}(t) = \begin{pmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{pmatrix}$$

$\mathbf{B}(t)$ is also a vector and in general may be written as

$$\mathbf{B}(t) = (B_x(t) \quad B_y(t) \quad B_z(t))$$

If we consider only the applied static magnetic field, we may write:

$$\mathbf{B} = (0 \quad 0 \quad B_0)$$

This expression indicates that the magnetic field has no time-dependence and is aligned along the z-direction.

In the more general sense we might include additional fields due to RF pulses, in this case we would write

$$\mathbf{B} = (B_1 \cos(\omega t) \quad -B_1 \sin(\omega t) \quad B_0)$$

the explicit time-dependence of the B_1 field (RF) is included via the trigonometric factors.

The *relaxation operator*, R , accounts for the propensity of the magnetic moment to reestablish its equilibrium polarization subsequent to its perturbation by an external source.

The general form of R may be written equivalently as a *relaxation matrix* as shown below:

$$\mathbf{R} = \begin{pmatrix} m_{xx} & m_{xy} & m_{xz} \\ m_{xy} & m_{yy} & m_{yz} \\ m_{zx} & m_{zy} & m_{zz} \end{pmatrix} = \begin{pmatrix} R_{xy} & 0 & 0 \\ 0 & R_{xy} & 0 \\ 0 & 0 & R_z \end{pmatrix} = \begin{pmatrix} 1/T_2 & 0 & 0 \\ 0 & 1/T_2 & 0 \\ 0 & 0 & 1/T_1 \end{pmatrix}$$

The nonzero elements of R correspond to the rates of decay/recovery of the macroscopic magnetic moment (polarization), which we observe to be a single exponential.

The relaxation rates parallel, $1/T_1$, and perpendicular, $1/T_2$, to the direction of the external magnetic field are distinct as per common observation.

Relaxation in the z-direction is named longitudinal relaxation or spin-lattice relaxation and the time-constant is T_1 .

Relaxation in the xy-plane is named transverse relaxation or spin-spin relaxation.

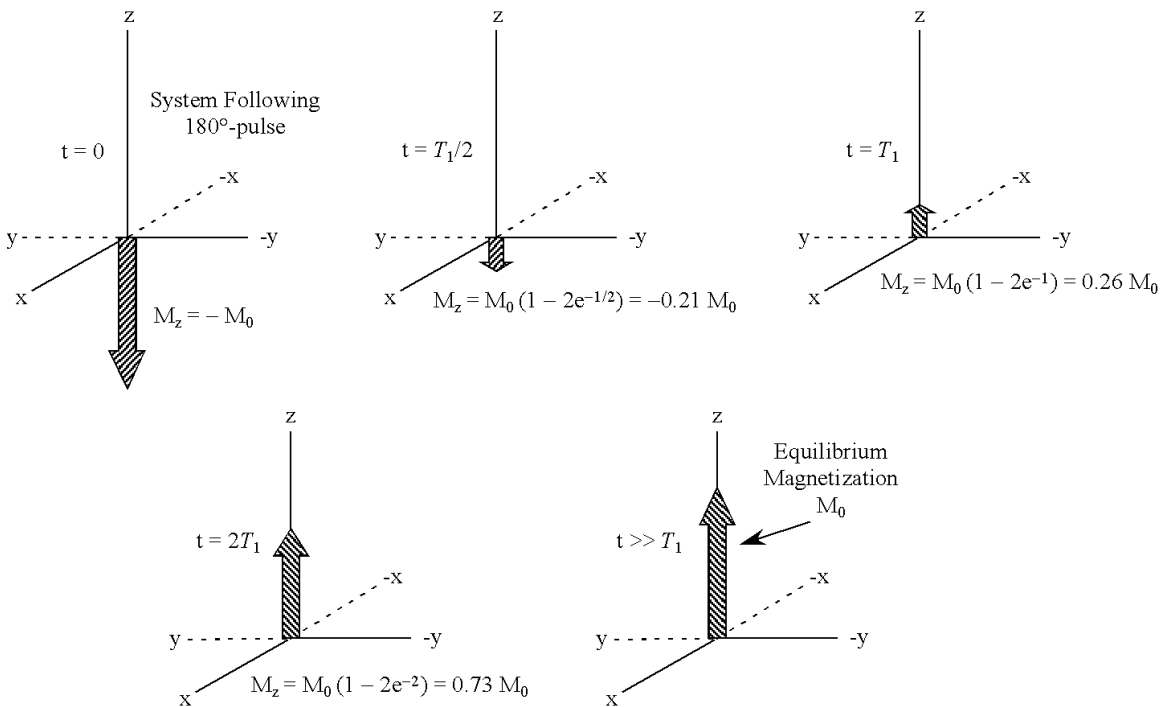
The matrix form of Bloch's equations may thus also be written as a series of differential equations:

$$\begin{aligned} \frac{d}{dt}M_x &= \gamma(M_y B_0 + M_z \sin(\omega t)) - \frac{M_x}{T_2} \\ \frac{d}{dt}M_y &= -\gamma(M_x B_0 + M_z \cos(\omega t)) - \frac{M_y}{T_2} \\ \frac{d}{dt}M_z &= -\gamma(M_x \sin(\omega t) - M_y \cos(\omega t)) - \frac{M_z - M_0}{T_1} \end{aligned}$$

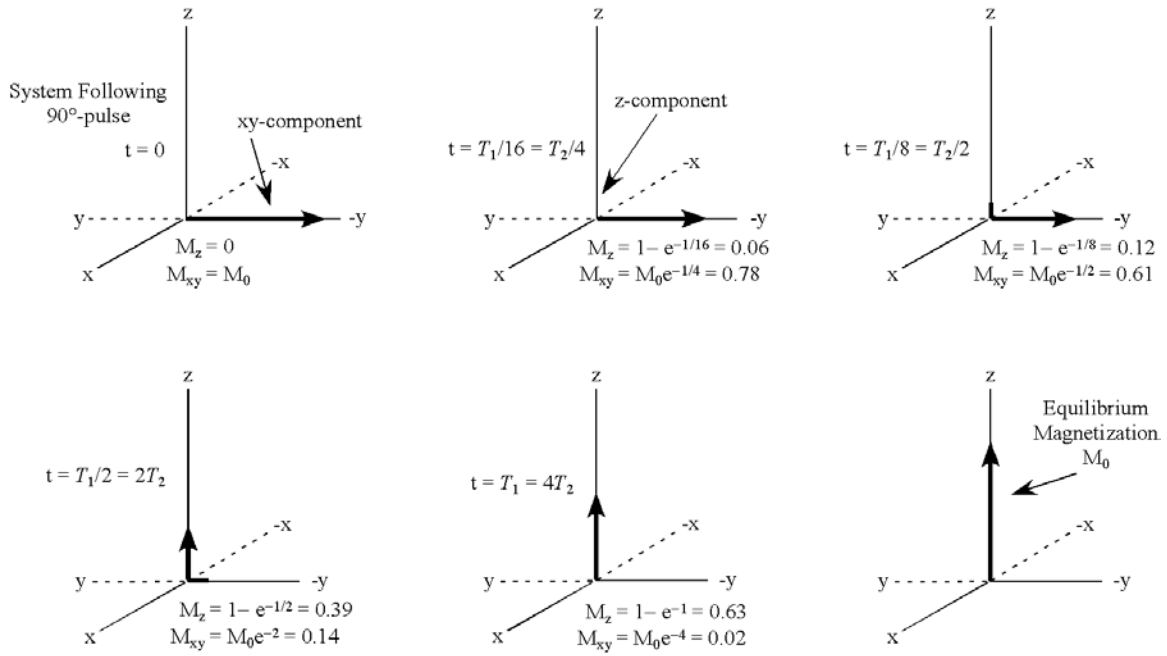
Note that Bloch's equations imply first-order relaxation, i.e., the recovery from perturbation is single-exponential.

We can now reconsider the phenomena of relaxation again we a more quantitative view:

Longitudinal Relaxation



Transverse Relaxation



Relaxation Mechanisms

Investigation of the *mechanism of relaxation* requires consideration of the origin of internal sources of fluctuating EM fields.

The most common origin of internal EM fields arises from the modulation of the dipolar interaction.

The classical interaction energy for a pair of magnetic dipoles is given by the following expression:

$$E = \frac{\vec{\mu}_I \cdot \vec{\mu}_S}{r_{IS}^3} - 3 \frac{(\vec{\mu}_I \cdot \vec{r}_{IS}) \cdot (\vec{\mu}_S \cdot \vec{r}_{IS})}{r_{IS}^5}$$

wherein \vec{r}_{IS} is the vector joining $\vec{\mu}_I$ and $\vec{\mu}_S$.

The QM version of this expression is obtained by making the following substitution:

$$\vec{\mu}_I = \hbar \gamma_I \vec{I}; \quad \vec{\mu}_S = \hbar \gamma_S \vec{S}$$

The energy of the interaction in the QM formalism is expressed as a Hamiltonian:

$$H_{dd} = \frac{1}{2} \frac{\gamma_I \gamma_S}{r_{IS}^3} (1 - 3 \cos^2 \theta) (3 I_z S_z - \vec{I} \cdot \vec{S})$$

wherein θ is the angle between r and the applied magnetic field (assume z-direction).

The time-dependent version of the Schrödinger Equation indicates the time-evolution of the spin systems is prescribed by the details of the Hamiltonian operator:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

Note that the interaction energy depends upon the cube of the internuclear distance and the orientation between the two spins and the applied magnetic field.

As the molecule tumbles in solution, the magnitude of the vector that between the two dipoles remains constant, but the orientation will clearly change, leading to a modulation of the energy.

The manner in which the orientation changes in time may be analyzed by considering the projection of a vector in the molecular frame over time. The expression that accounts for this behavior is called an *autocorrelation function* (or simply the correlation function), and is usually written as $G(\tau)$.

The exact form of the correlation depends upon the nature of the motion. For Brownian motion $G(\tau)$ may be suitably modeled as an exponential decay. The correlation function may thus be written as:

$$G(\tau) = e^{-t/\tau}$$

The rate of tumbling for a spherical molecule is characterized by a *correlation time*, τ that is defined as follows:

$$\tau = \frac{\eta V_{eff}}{k_b T}$$

wherein η is the solution viscosity, V_{eff} is the effective molecular volume, k_b is Boltzmann's constant and T is the temperature in K.

The Fourier transform of the correlation function tells us about the frequency distribution the motions and is called the *spectral density function* and is usually written as $J(\omega)$.

The Fourier transform of an exponential function is a *Lorentzian function*:

$$J(\omega) = \frac{C\tau}{1 + \omega^2\tau^2}$$

The value of the spectral density thus depends both on the transition frequency and the correlation time:

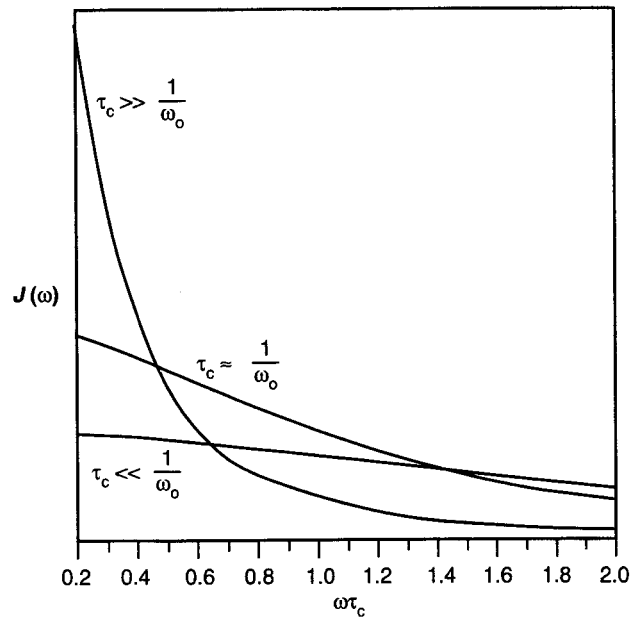


Figure 8.1 from Becker, E.D. (2000) High Resolution NMR. Academic Press, New York

Notice that for $\omega\tau \ll 1$ that the value of $J(\omega)$ decreases rapidly as a function of $\omega\tau$, whereas for $\omega\tau \gg 1$ the opposite is true.

In fact, in the limit as $\omega\tau \gg 1$, $J(\omega)$ is directly proportional to τ .

Transition Rates

Perturbation theory provides us with an expression generally known as *Fermi's golden rule*:

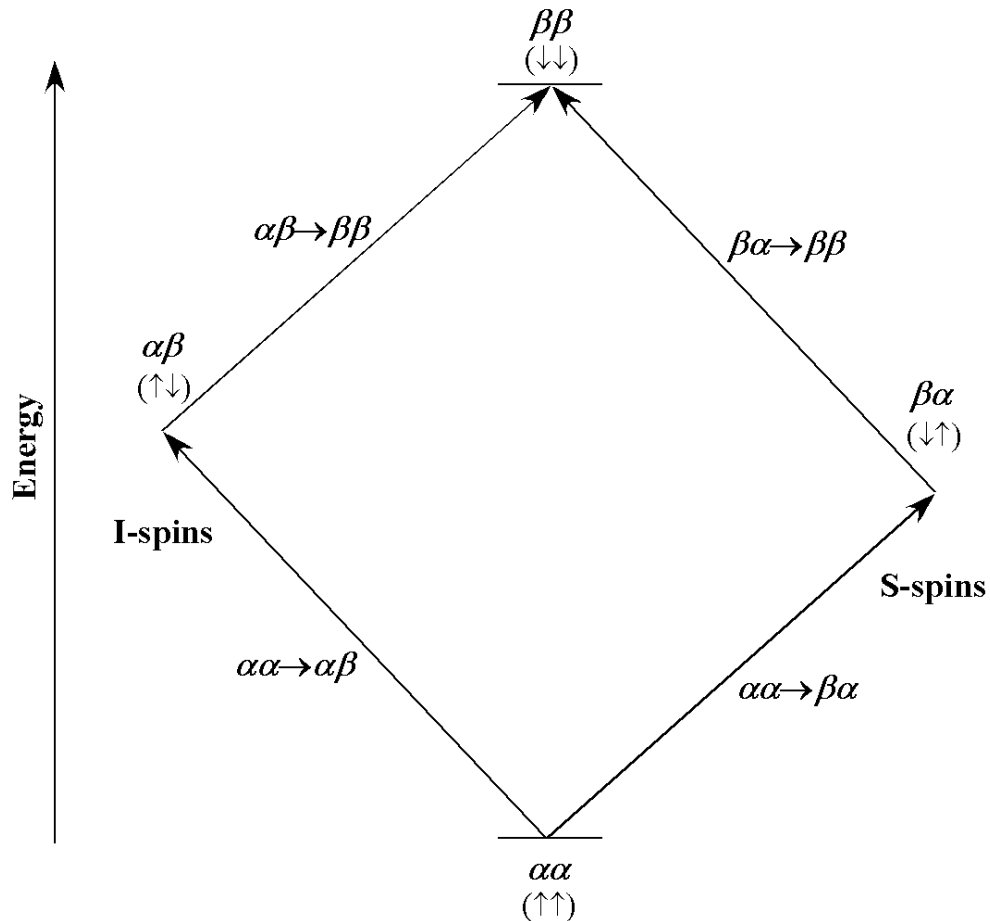
$$W_{i \rightarrow j} \equiv \text{transition probability per unit time}$$

$$= \frac{1}{\hbar^2 t} \int_0^t \int_0^t \langle i | H(t') | j \rangle \langle j | H(t'') | i \rangle e^{-i\omega_{ij}(t'-t'')} dt' dt''$$

wherein ω_{ij} is the transition frequency (rad s⁻¹).

The expression is the Fourier transform of the Hamiltonian composed with itself and evaluated over time.

The W_{ij} are defined with respect to our energy level diagram:



$W_{i \rightarrow j} \equiv$ transition probability per unit time

$$= \frac{1}{\hbar^2 t} \int_0^t \int_0^t \langle i | H(t') | j \rangle \langle j | H(t'') | i \rangle e^{-i\omega_{ij}(t'-t'')} dt' dt''$$

This expression is entirely general and may be used to predict the transition rates due to RF irradiation:

$$W_{i \rightarrow j} = 2\pi \gamma_I \gamma_S B_1^2 \langle i | H_{RF}(t) | j \rangle^2 \delta(\omega_{ij} - \omega_1)$$

wherein $\delta(\omega_{ij} - \omega_1)$.

This is the QM expression of how RF fields stimulate emission in the NMR spin system.

As we now know, the energy of dipolar interaction varies over time due to molecular tumbling, and this variation also produces transitions in the energy levels.

If we substitute the dipolar coupling Hamiltonian into the transition rate expression, we will obtain the following:

$$W_0 = W(\alpha\beta \leftrightarrow \beta\alpha) \propto \frac{\gamma_I \gamma_S}{r_{IS}^6} J(\omega_I - \omega_S) \approx \frac{\gamma_I \gamma_S}{2r_{IS}^6} J(0) \text{ when } I=S$$

$$W_{1I} = W(\alpha\alpha \leftrightarrow \beta\alpha) = W(\alpha\beta \leftrightarrow \beta\beta) \propto \frac{\gamma_I \gamma_S}{r_{IS}^6} J(\omega_I)$$

$$W_{1S} = W(\alpha\alpha \leftrightarrow \alpha\beta) = W(\beta\alpha \leftrightarrow \beta\beta) \propto \frac{\gamma_I \gamma_S}{r_{IS}^6} J(\omega_S)$$

$$W_2 = W(\alpha\alpha \leftrightarrow \beta\beta) \propto \frac{\gamma_I \gamma_S}{r_{IS}^6} J(\omega_I + \omega_S)$$

We may write the rates of longitudinal and transverse relaxation in term of these transition rates to obtain the following expressions:

$$R_1^H(^1\text{H}, ^{13}\text{C}) = \frac{1}{T_1} = \frac{C\gamma_H^2\gamma_C^2}{r_{\text{HC}}^6} [J(\omega_H - \omega_C) + 2J(\omega_H) + J(\omega_H + \omega_C)]$$

$$R_1^C(^1\text{H}, ^{13}\text{C}) = \frac{1}{T_1} = \frac{C\gamma_H^2\gamma_C^2}{r_{\text{HC}}^6} [J(\omega_H - \omega_C) + 2J(\omega_C) + J(\omega_H + \omega_C)]$$

$$R_1^H(^1\text{H}, ^1\text{H}) = \frac{1}{T_1} = \frac{C\gamma_H^4}{r_{\text{HH}}^6} [J(\omega_H) + 4J(2\omega_H)]$$

$$R_2^H(^1\text{H}, ^{13}\text{C}) = \frac{1}{T_2} = \frac{C\gamma_H^2\gamma_C^2}{2r_{\text{HC}}^6} [4J(0) + J(\omega_H - \omega_C) + 3J(\omega_H) + 6J(\omega_C) + 6J(\omega_H + \omega_C)]$$

$$R_2^C(^1\text{H}, ^{13}\text{C}) = \frac{1}{T_2} = \frac{C\gamma_H^2\gamma_C^2}{2r_{\text{HC}}^6} [4J(0) + J(\omega_H - \omega_C) + 3J(\omega_C) + 6J(\omega_H) + 6J(\omega_H + \omega_C)]$$

$$R_2^H(^1\text{H}, ^1\text{H}) = \frac{1}{T_2} = \frac{C\gamma_H^4}{2r_{\text{HH}}^6} [3J(0) + 5J(\omega_H) + 2J(2\omega_H)]$$

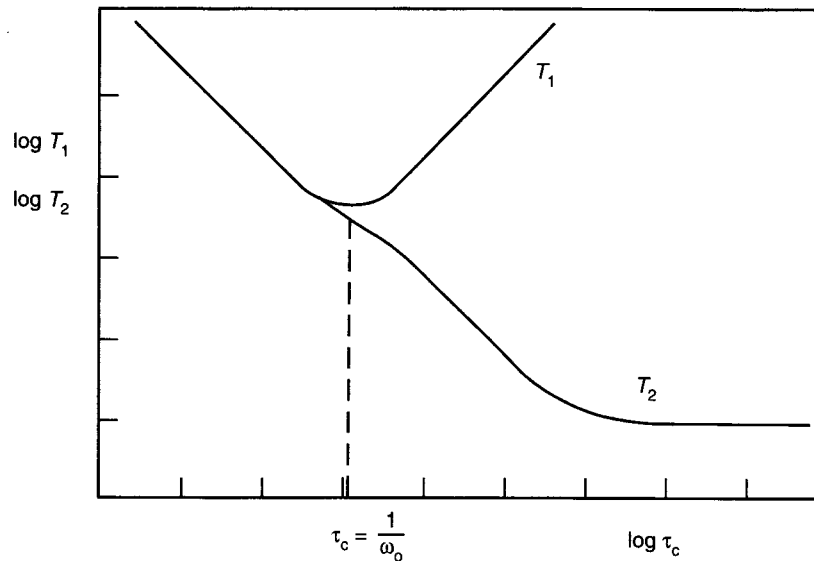


Figure 8.2 from Becker, E.D. (2000) High Resolution NMR. Academic Press, New York.